D8 – Phénomènes de transport

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Niveau : L2

Prérequis

- Mécanique Lagrangienne
- Problème à deux corps
- Mécanique du solide

Expériences

- Conservation de l'énergie et de la charge $v_z + gt$ durant la chute d'une réglette.

Table des matières

D2) Lois de Conservation en Dynamique

L3 - moindre action, eq E-L, pb à 2 corps, meca solide

I Symétrie et Conservation

1) Non unicité du Lagrangien

Soit $F(q, t)$

$$L'(\dot{q}, q, t) = L(\dot{q}, q, t) + \frac{dF}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}\right) - \cancel{\frac{\partial L}{\partial q_i}} - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$$

Or $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\frac{dq}{dt} = \frac{\partial F}{\partial t} + \dot{q}\frac{\partial F}{\partial q_i}$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \cancel{\frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)} - \cancel{\frac{\partial}{\partial q_i}\frac{dF}{dt}} = 0$$

$$\Rightarrow \boxed{L' = L + \frac{dF}{dt} \quad \text{décrit aussi le système}}$$

2) Théorème de Noether

Soit une transformation des coord $q_i \to q_i'$

avec $q_i' = q_i + \varepsilon g_i(q_i, t) = q_i + \delta q_i$

$\dot{q}_i' = \dot{q}_i + \varepsilon \frac{\partial g_i}{\partial q_j}\dot{q}_j + \varepsilon \frac{\partial g_i}{\partial t} = \dot{q}_i + \delta \dot{q}_i$

~~On définit~~ $\cancel{L(\dot{q}_i', q_i', t) = L(q}$

Cette transfo décrit une symétrie infinitésimale si et seulement si :

$$\boxed{\mathcal{L}(\dot{q}_i', q_i', t) \underset{\varepsilon \ll 1}{=} \mathcal{L}(\dot{q}_i, q_i, t) + \varepsilon \frac{dF}{dt}(q_i, t)}$$

$\boxed{\text{HM}}$ $\mathcal{L}(\dot q_i + \delta\dot q_i, q_i + \delta q_i, t) - \mathcal{L}(\dot q_i, q_i, t) = \varepsilon \frac{dF}{dt}$

$$\sum_i \left(\delta\dot q_i \frac{\partial \mathcal L}{\partial \dot q_i} + \delta q_i \frac{\partial\mathcal L}{\partial q_i}\right) = \varepsilon\frac{dF}{dt}$$

$$\sum_i \left[\cancel{\delta q_i\left(\frac{\partial\mathcal L}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal L}{\partial\dot q_i}\right)} + \delta q_i \frac{d}{dt}\frac{\partial\mathcal L}{\partial \dot q_i} + \delta\dot q_i\frac{\partial\mathcal L}{\partial\dot q_i}\right] = \varepsilon\frac{dF}{dt}$$

$$\sum_i \frac{d}{dt}\left(\delta q_i \frac{\partial \mathcal L}{\partial \dot q_i}\right) = \varepsilon\frac{dF}{dt}$$

$$\boxed{\frac{d}{dt}\left(\sum_i \varepsilon\, g_i(q_j,t)\frac{\partial\mathcal L}{\partial\dot q_i} - \varepsilon F\right) = 0}$$

$$Q = \sum_i \left(g_i(q_j,t)\frac{\partial\mathcal L}{\partial \dot q_i}\right) - F(q_j,t)$$

Q est une charge de Noether, c'est une constante de mouvement.

3) Conservations usuelles

Cas particulier : $\frac{\partial\mathcal L}{\partial q_k} = 0$

On déf la transfo $\begin{cases} g_k(q,t) = 1 \\ g_{j\neq k}(q,t) = 0\end{cases}$

$\delta q_k = \varepsilon \qquad \delta\dot q_k = 0$

$\delta q_{j\neq 0} = 0 \qquad \delta \dot q_{j\neq 0} = 0$

$\rightarrow \delta\dot q_i \frac{\partial\mathcal L}{\partial\dot q_i} + \delta q_i\frac{\partial \mathcal L}{\partial q_i} = \varepsilon\frac{\partial\mathcal L}{\partial q_k} = 0$ → on choisit $F = 0$

$$Q = \sum_i g_i(q_j,t)\frac{\partial\mathcal L}{\partial\dot q_i} = \boxed{\frac{\partial \mathcal L}{\partial \dot q_k} = \text{const}}$$

Si q_k est une coord spatiale → conservation de l'impulsion

Si q_k est une coord angulaire → conservation du moment cinétique

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} = |\vec{r} \wedge \vec{p}|$$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{\vec{p}}^2 \right) - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\vec{p}}} = m \vec{v} = \vec{p}$$

Invariance dans le temps

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

Alors $\displaystyle \frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \dot{q}_i \frac{\partial \mathcal{L}}{\partial q_i} + \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$$= 0 + \cancel{\dot{q}_i \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)} + \dot{q}_i \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt} \left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$

$$\frac{d}{dt} \left(\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right) = 0$$

→ $\boxed{H \equiv \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}}$ est conservée

$$\left(H = \dot{q}_i \, m \dot{q}_i - \left(\frac{1}{2} m \dot{q}_i^2 - V(q) \right) = \frac{1}{2} m \dot{q}_i^2 + V(q) \right.$$

Exemple gravitation

$$\mathcal{L} = \frac{1}{2} m \dot{z}^2 - mgz$$

$$\frac{1}{2} m \dot{z}^2 + mgz = H$$

$$\dot{z}^2 = 2\left(\frac{H}{m} - gz\right)$$

$$\dot{z} = \sqrt{2\left(\frac{H}{m} - gz\right)}$$

$$\delta z = \varepsilon \quad ; \quad \delta \dot{z} = 0$$

$$\mathcal{L}(\dot{z} + \delta \dot{z}, z + \delta z, t) = \frac{1}{2} m \dot{z}^2 - mg(z + \delta z)$$

$$\hookrightarrow \delta \mathcal{L} = -mg\,\delta z = -mg\varepsilon$$

$$F(z,t) \equiv -mgt$$

$$Q = \dot{z} \frac{\partial \mathcal{L}}{\partial \dot{z}} - F = m \dot{z}^2 + mgt$$

$$\boxed{\dot{z} + gt = \text{Cst}}$$

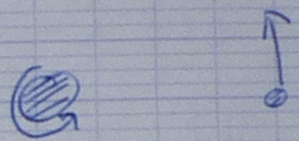
$$\dot{z} = \text{Cst} - gt$$

$$z = z_0 + v_{z_0} t - \frac{1}{2} g t^2$$

$$\begin{aligned} H &= \frac{1}{2} m \dot{z}^2 + mgz \\ &= \frac{1}{2} m (v_0 - gt)^2 + mg\left(z_0 + v_0 t - \frac{1}{2} g t^2\right) \\ &= \frac{1}{2} m \left(v_0^2 + \cancel{g^2 t^2} - \cancel{2 v_0 g t}\right) + mg\left(z_0 + \cancel{v_0 t} - \cancel{\frac{1}{2} g t^2}\right) \\ &= \frac{1}{2} m v_0^2 + mg z_0 \end{aligned}$$

II Application : marées

Système Terre Lune



$$E_c = \frac{1}{2} J_\oplus \omega_\oplus^2 + \frac{1}{2} J_☾ \omega_☾^2 + \frac{1}{2} m_\oplus \left(\frac{m_☾}{m_\oplus + m_☾} D\right)^2 \Omega^2$$

$$+ \frac{1}{2} m_☾ \left(\frac{m_\oplus}{m_\oplus + m_☾} D\right)^2 \Omega^2$$

Loi Kepler $\boxed{\Omega = \sqrt{\frac{G m_\oplus + m_☾}{D^3}}}$

$$\hookrightarrow E_c = \frac{1}{2} J_\oplus \omega_\oplus^2 + \frac{1}{2} J_☾ \omega_☾^2 + \frac{1}{2} \frac{m_☾ m_\oplus}{\cancel{m_\oplus + m_☾}} \cancel{D^2} \frac{G \cancel{m_\oplus + m_☾}}{D^{\cancel{3}}}$$

$$\boxed{E_c = \frac{1}{2} J_\oplus \omega_\oplus^2 + \frac{1}{2} J_☾ \omega_☾^2 + \frac{1}{2} \frac{G m_\oplus m_☾}{D}}$$

$$\boxed{\mathcal{L} = E_c - E_p = \frac{1}{2} J_\oplus \omega_\oplus^2 + \frac{1}{2} J_☾ \omega_☾^2 + \frac{3}{2} \frac{G m_\oplus m_☾}{D}}$$

$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow$ conservation du moment cinétique L

$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow$ conservation de H

$$H = \frac{1}{2} J_\oplus \omega_\oplus^2 + \frac{1}{2} J_☾ \omega_☾^2 - \frac{1}{2} \frac{G m_\oplus m_☾}{D}$$

$$L = J_\oplus \omega_\oplus + J_☾ \omega_☾ + \frac{m_\oplus m_☾}{m_\oplus + m_☾} D^2 \Omega$$

$$L = J_\oplus \omega_\oplus + J_☾ \omega_☾ + m_\oplus m_☾ \sqrt{\frac{G D}{m_\oplus + m_☾}}$$

On introduit les forces microscopiques

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_{micro}$$

$$\frac{d\mathcal{L}'}{\partial t} = 0 \quad \text{(les forces micro sont conservative)}$$

→ Energie U est conservée.

$$J = \gamma R^2 M$$

$$J_{\text{☾}} = \frac{\gamma_{\text{☾}}}{\gamma_{\oplus}} \frac{R_{\text{☾}}^2 M_{\text{☾}}}{R_{\oplus}^2 M_{\oplus}}$$

$$J_{\text{☾}} = J_{\oplus} \frac{\gamma_{\text{☾}}}{\gamma_{\oplus}} \frac{1}{1090}$$

$$dU = 0 = TdS + dH$$

$$dH = -TdS \leqslant 0 \quad \text{(second principe)}$$

↳ $\boxed{\frac{dH}{dt} \leqslant 0}$

$$H = \frac{1}{2} J_{\oplus} \omega_{\oplus}^2 + \cancel{\frac{1}{2} J_{\text{☾}} \omega_{\text{☾}}^2} - \frac{1}{2} \frac{G M_{\oplus} M_{\text{☾}}}{D} \qquad \frac{dH}{dt} \leqslant 0$$

$$L = J_{\oplus} \omega_{\oplus} + \cancel{J_{\text{☾}} \omega_{\text{☾}}} + M_{\oplus} M_{\text{☾}} \sqrt{\frac{GD}{M_{\oplus} + M_{\text{☾}}}} \qquad \frac{dL}{dt} = 0$$

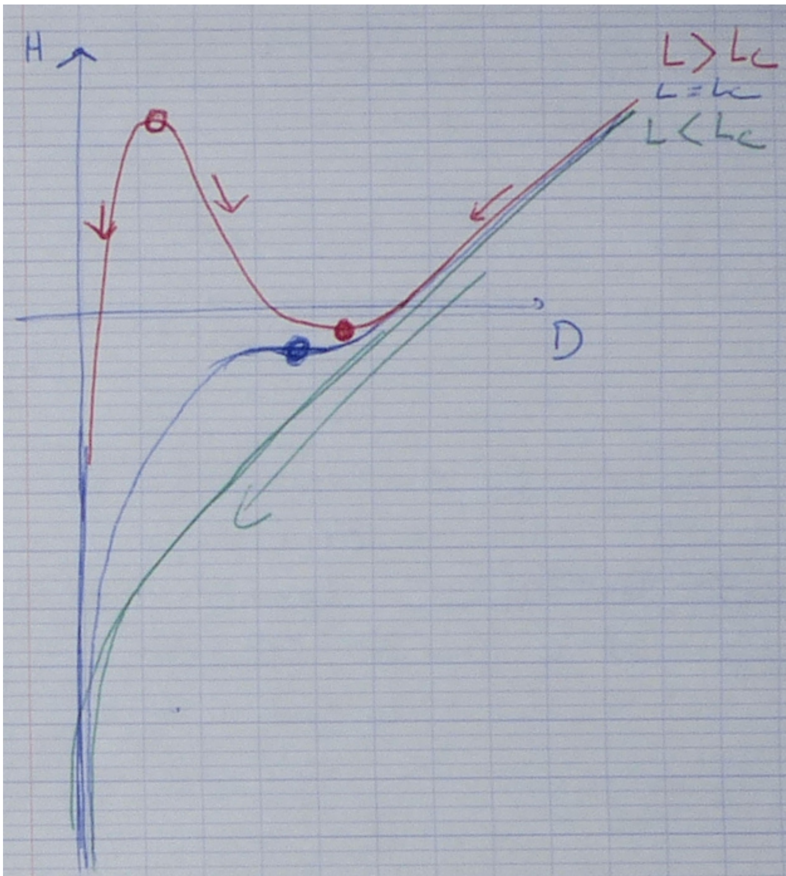
$$\boxed{\omega_{\oplus} = \frac{L}{J_{\oplus}} - \frac{M_{\oplus} M_{\text{☾}}}{J_{\oplus}} \sqrt{\frac{GD}{M_{\oplus} + M_{\text{☾}}}}}$$

$$\rightarrow H = \frac{1}{2 J_{\oplus}} \left[L^2 + \frac{M_{\oplus}^2 M_{\text{☾}}^2}{M_{\oplus} + M_{\text{☾}}} GD - 2 L M_{\oplus} M_{\text{☾}} \sqrt{\frac{GD}{M_{\oplus} + M_{\text{☾}}}} \right] - \frac{1}{2} \frac{G M_{\oplus} M_{\text{☾}}}{D}$$

$$\rightarrow \frac{dH}{dD} = \frac{M_{\oplus}^2 M_{\text{☾}}^2 G}{2 J_{\oplus} (M_{\oplus} + M_{\text{☾}})} - \frac{L M_{\oplus} M_{\text{☾}}}{2 J_{\oplus}} \sqrt{\frac{G}{D (M_{\oplus} + M_{\text{☾}})}} + \frac{G M_{\oplus} M_{\text{☾}}}{2 D^2}$$

$$\frac{dH}{dD} = \frac{M_{\oplus} M_{\text{☾}}}{2} \sqrt{\frac{G}{D (M_{\oplus} + M_{\text{☾}})}} \left[\frac{M_{\oplus} M_{\text{☾}}}{J_{\oplus}} \sqrt{\frac{DG}{M_{\oplus} + M_{\text{☾}}}} - \frac{L}{J_{\oplus}} + \sqrt{\frac{G (M_{\oplus} + M_{\text{☾}})}{D^3}} \right]$$

$$\boxed{\frac{dH}{dD} = \frac{M_{\oplus} M_{\text{☾}}}{2} \sqrt{\frac{G}{D (M_{\oplus} + M_{\text{☾}})}} \left(\Omega - \omega_{\oplus} \right)}$$



Application

$M_L = 0{,}0123\ M_\oplus$ $\qquad J_\oplus = 0{,}3\ R_\oplus^2 M_\oplus$

$R_\oplus = 6378$ km $\qquad M_\oplus = 5{,}974\ 10^{24}$ kg $\qquad$ ans =

$\omega_\oplus = 2301$ rad/ans

$\bar{D} = 60{,}27\ R_\oplus$ $\qquad G = 1{,}53\ 10^{9}\ R_\oplus^3 M_\oplus^{-1} \text{an}^{-2}$

$L = 4403\ R_\oplus^2\ \text{ans}^{-1}$ $\qquad \bar{\Omega} = 84{,}11$ rad/ans

$\frac{dH}{dD} = -68275\ M_\oplus R_\oplus\ \text{ans}^{-2} = -2{,}61\ 10^{21}$ J/m

$\frac{d\bar{D}}{dt} = 0{,}038$ m/ans $\qquad \frac{dH}{dt} = \frac{dH}{dD}\frac{d\bar{D}}{dt} = -9{,}93\ 10^{19}$ J/ans

$$\boxed{\frac{dH}{dt} = -3{,}15 \text{ TJ/s}}$$

Beaucoup ? Oui, mais non

Activité humaine en 2019 = 18,5 TJ/s $\qquad$ USA = 2,9 TW

Points equilibre

$D = 2,168\ R_\oplus$ $\Omega = 12329$ rad/ans

↳ 1961,2 jour/ans → jour de 4h28

$D = 84,2\ R_\oplus$ $\Omega = 50,94$ rad/ans

↳ 7,1 jours/ans → jour de 51d 9h22

(1233h22)

On estime la position initial de la lune à $3,8\ R_\oplus$

→ $\Omega = 5313$ rad/ans → 844,6 pleine lune par ans.

↳ Une toute les 10h23

$\omega_\oplus = 11570$ rad/ans → 1840,4 jours/ans

↳ jours de 4h46

$\omega - \Omega = 6257$

$\Omega = 5313$

$\omega = 6257 + 5313$